M.U

QNo→ Let N and N^* be normed linear spaces and T is a linear transformation of N into N^*. Then the following statements are equivalent:-

(i) T is Continuous,

(ii) T is Continuous at the origin.

Prove this.

or, QNo→ Let N and N^* be normed linear space and T a linear transformation of N into N^*. Then the following Condition are equivalent to one another:-

(i) T is Continuous,

(ii) T is Continuous at the origin.

[i.e., $x_n \to 0 \Rightarrow T(x_n) \Rightarrow 0$].

(iii) T is bounded.

[i.e., ∃ a real number $K \geq 0$ s.t.

$\|T(x)\| \leq K\|x\| \; \forall \; x \in N$]

(iv) If $S = \{x : \|x\| = 1\}$ is Closed sphere in N, then its image T(S) is bounded set N^*.

Proof:- ~~First~~ (i) ⇒ (ii)

Let $T : N \to N^*$ be a Continuous map and $\langle x_n \rangle$ be a sequence in N such that

$\underset{n\to\infty}{Lt.}\, x_n = 0$

By Continuity of T,

$x_n \to 0 \Rightarrow T(x) \to T(0)$. But $T(0) = 0$ for any linear map.

$\Rightarrow T(x) \to 0$.

This proves that T is continuous at origin.

Now, we claim (ii) $\Rightarrow$ (i)

Let T be continuous at the origin.

Let $<x_n>$ be a sequence in N such that.

$$\lim_{n\to\infty} x_n = x \in N.$$

Then, $x_n \to x \Rightarrow x_n - x \to 0 \Rightarrow T(x_n - x) \to T(0) = 0$.

[For T is continuous at origin].

$\Rightarrow T(x_n) - T(x) \to 0$

$\Rightarrow T(x_n) \to T(x)$.

Thus, $x_n \to x \Rightarrow T(x_n) \to T(x)$.

By defn this proves that T is continuous.

(II) To prove (ii) $\Rightarrow$ (iii). Let T be continuous at $x=0$. To prove (iii), Then T is not bounded. Then for each +ve integer n, we can find vector $x_n \in N$ such that,

$$\|T(x_n)\| > n\|x_n\|$$

$$\text{or, } \frac{\|T(x_n)\|}{n\|x_n\|} > 1 \quad \text{--- (1)}$$

$$\text{or, } \left\|T\left[\frac{x_n}{n\|x_n\|}\right]\right\| = \frac{\|T(x_n)\|}{n\|x_n\|} > 1.$$

Taking $y_n = \dfrac{x_n}{n\|x_n\|}$, we get $T(y_n) > 1$ --- (2)

Also, $\|y_n\| = \left\| \frac{x_n}{n\|x_n\|} \right\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n}$.

$$\therefore \lim_{n\to\infty} \|y_n\| = \frac{1}{\infty} \quad \text{— (6)}$$

$$\therefore \|y_n\| = 0 \quad \text{— (3)}$$

$\therefore \lim_{n\to\infty} y_n = 0$. But $\lim_{n\to\infty} T(y_n) > 1$, by (2)

Finally, $y_n \to 0 \Rightarrow T(y_n)$ does not tend to zero as $n \to \infty$.

A ~~Continuous~~ Contradiction. For T is Continuous at $x = 0$.

Hence (iii) is true.

To Prove (iii) $\Rightarrow$ (ii).

Let T be bounded so that $\exists$ a real no. $K \geq 0$.

Such that, $\|T(x)\| \leq K\|x\| \ \forall x \in N$ — (4)

Let $<x_n>$ be a sequence in N such that,

$$\lim_{n\to\infty} x_n = 0 \in N.$$

$$\Rightarrow \|x_n\| \to 0 \text{ as } n \to \infty \quad \text{— (5)}$$

For norm is a Continuous map.

By (4), $\|T(x_n)\| \leq K\|x_n\|$.

using (5), we find that $\|T(x_n)\| \to 0$ as $n \to \infty$.

$\Rightarrow T(x_n) \to 0$ as $n \to \infty$.

This shown that T is Continuous at the origin.

(III) To Prove (iii) $\Rightarrow$ (iv)

Let (iii) be true, i.e., let $\exists\ K \geq 0$ such that

$$\|T(x)\| \leq K\|x\| \ \forall\ x \in N \quad — (6)$$

$$(6) \Rightarrow \|T(x)\| \leq K\|x\| \ \forall\ x \in S \subset N \quad — (7)$$

But, $x \in S \Rightarrow T(x) \in T(S)$.

Hence, (7) proves that $T(S)$ is bounded.

To Prove (iv) $\Rightarrow$ (iii). Let (iv) be true, then $\exists\ K > 0$ such that, $\|T(x)\| \leq K\|x\| \ \forall\ x \in S$ — (8)

If $x = 0$, then $T(x) = 0$ So, that (8) is true for $x = 0$ — (9).

If $x \neq 0$, then put $y = \frac{x}{\|x\|}$ so that,

$$\|y\| = \frac{\|x\|}{\|x\|} = 1,$$

Consequently $x \in S$ and so by (8) $\|T(y)\| \leq K\|y\|$

$$\text{or, } \left\| T\left(\frac{x}{\|x\|}\right)\right\| \leq K\left\|\frac{x}{\|x\|}\right\| = \frac{K\|x\|}{\|x\|} = K$$

$$\text{or, } \|T(x)\| \leq K \ \forall\ x \neq 0 \quad — (10)$$

From (9) & (10), we have

$$\|T(x)\| \leq K\|x\| \ \forall\ x \in N.$$

$\Rightarrow T$ is bounded. Hence (iii)

This completes the proof.